Hamburg Lectures on Spectral Networks 2018, Lecture 3

I didn't get to give the lecture.

1. Definition + Construction Of Spectral Networks

(Physical interpretation: Solitons With Charge in $\Gamma(z, z)$)

2. Interfaces + Formal Parallel Transport

3. Abelianization + Nonabelianization + True Parallel Transport

4. Morphisms of Spectral Networks: BPS States

5. Comments on Applications

1. Definition & Construction Of Spectral Networks

Data for a (WKB) spectral network

Is a branched cover

$$\pi: \Sigma \to C \qquad \Sigma \subset T^*C$$

with at least one puncture on C where λ has a pole

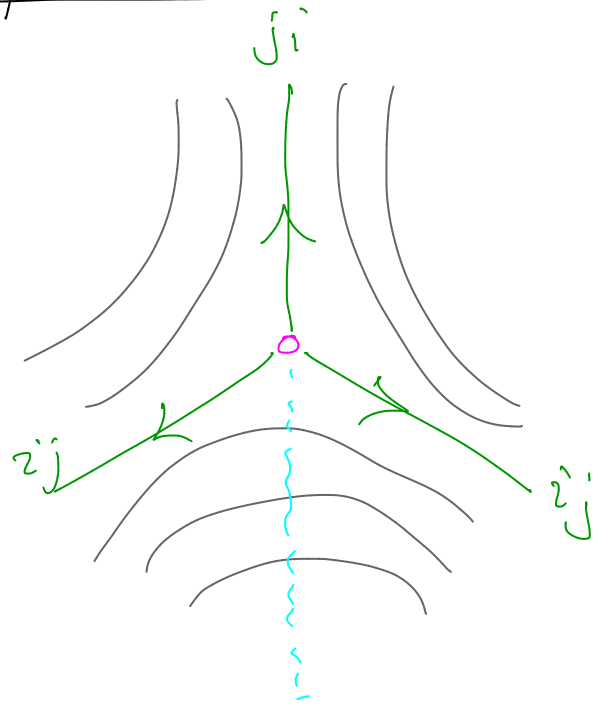
Def: A WKB curve of phase ϑ and type i,j is a solution to

$$\langle \partial_t, \lambda_i - \lambda_j \rangle = e^{i\vartheta}$$

in some open nbd on C for some pair of sheets i,j in a trivialization of the branched cover in that neighborhood.

Remarks:
- This defines a foliation of C
- Note the constant phase condition means that $\left| \int \lambda_i - \lambda_j \right| = \int |\lambda_i - \lambda_j|$ so is a length-minimizing condition

Behavior near a simple branch point of type (ij):

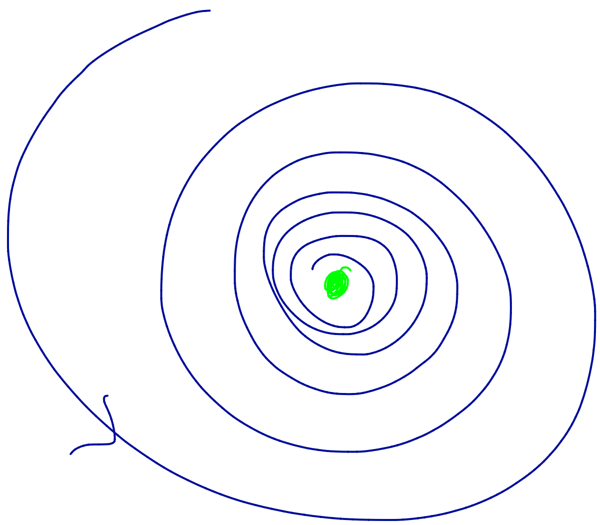


$$\int_{b}^{z} (\lambda_i - \lambda_j) \sim \frac{4}{3} z^{3/2} = e^{i\vartheta} t$$

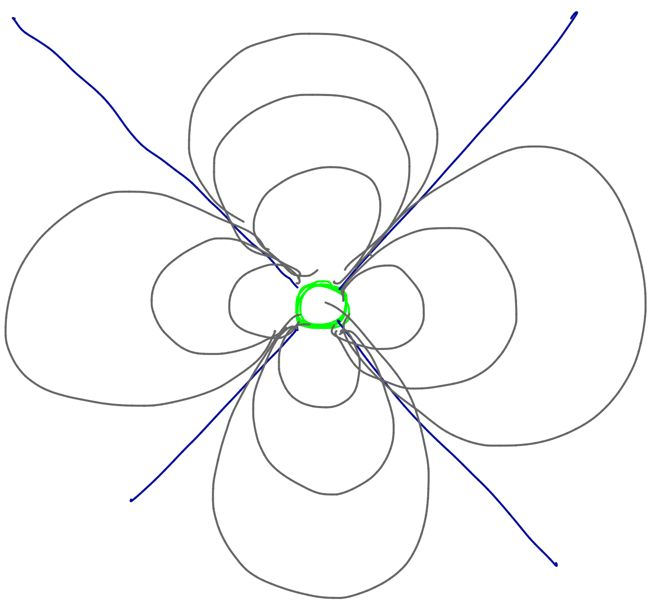
$$t \in \mathbb{R}_+$$

<u>Note:</u> Therefore as ϑ increases the WKB curves rotate CCW.

Behavior near a <u>regular singular point</u>:

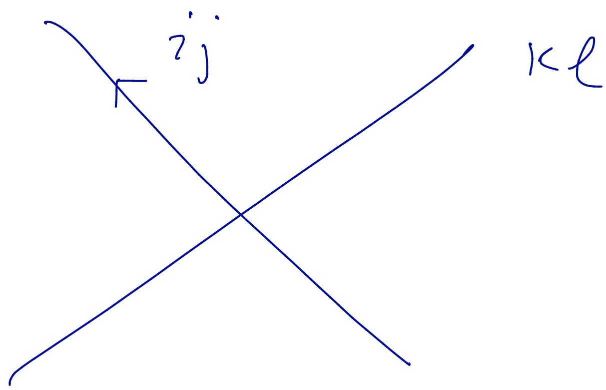


Behavior near an <u>irregular singular point</u>:

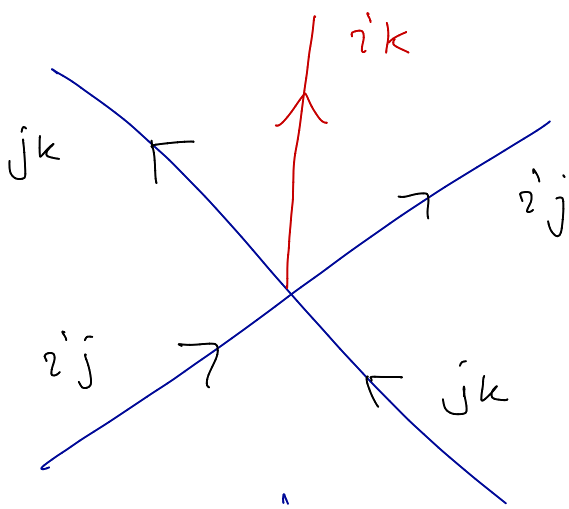


These singular points serve as attractors for the flow. So we can start the flows at branchpoints and then just evolve the differential equation smoothly, even though there is no global trivialization of the cover $\pi : \Sigma \to C$

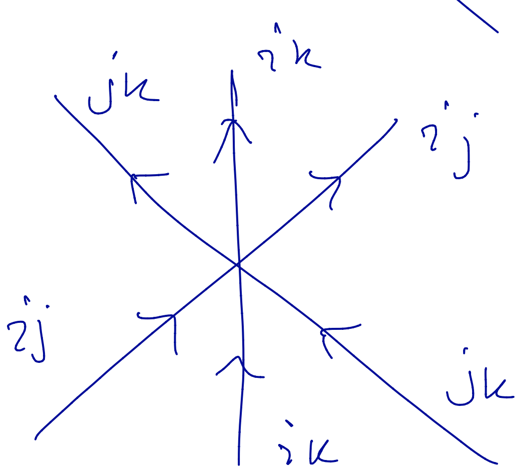
To evolve we need a rule for when curves collide



no sheets coincide



Sheet is born (or dies)



general case.

Now can evolve with cutoff

$$\left| \int_{z_0}^{z} \lambda \right| \leq \Lambda \qquad \Lambda \longrightarrow \infty$$

This produces movies see:

- Andy Neitzke homepage

http://web.ma.utexas.edu/users/neitzke/

Scroll down to Mathematica files and click on the Notebook for plotting spectral networks

- LOOM - By Pietro Longhi & Chan Park:

http://het-math2.physics.rutgers.edu/loom/

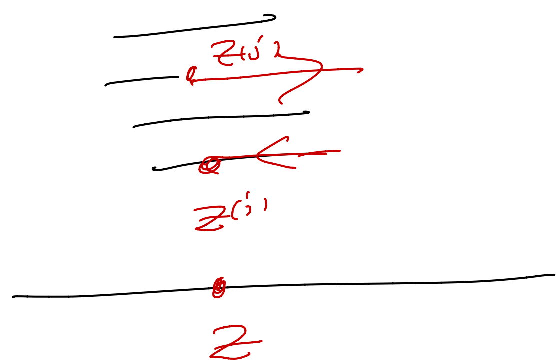
The result is a graph $W_\vartheta \subset C$ with segments labeled by an orientation and pair of sheets.

Remark: Physical Interpretation.

We mentioned above the zoo of BPS states - among them are the solitons on the 1+1 D Soliton defects $\mathbb{S}_z$, $z \in \mathbb{C}$.

Recall they have charges

$$\gamma_{ij} \in \Gamma_{ij}(z,z)$$



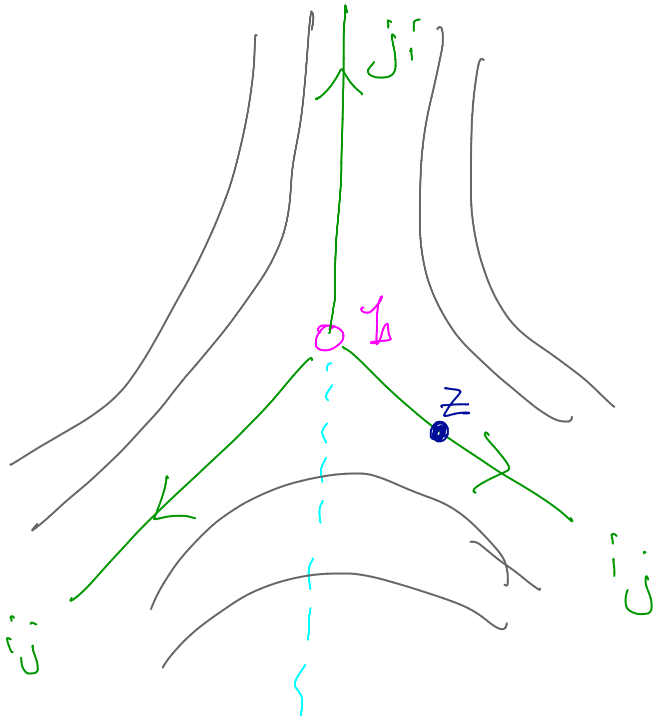
$N=2$ Central Charges:

$$\int_{\gamma_{ij}} \lambda$$

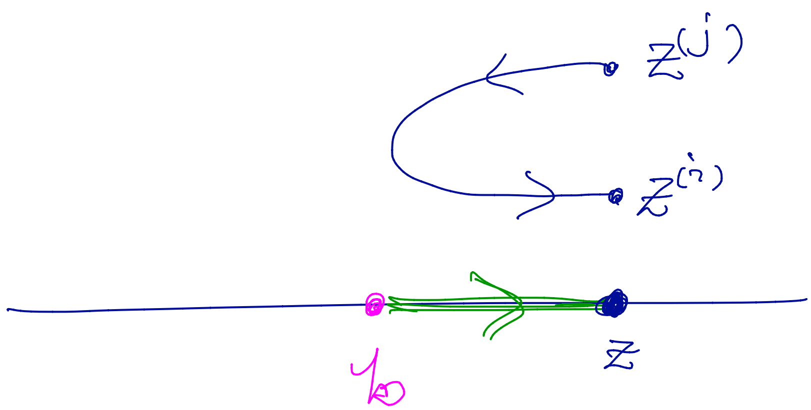
One can show that for phase $e^{i\vartheta}$

$$W_\vartheta = \{ z \mid \mathbb{S}_z \text{ has solitons of phase } \vartheta \}$$

In fact, near a simple branch point b of type (ij) we have



We always have the "simpleton":



For this charge $\mu(\gamma_{ij}) = 1$.

The remaining BPS degeneracies then follow from wall-crossing.

In general $\mu(\gamma_{ij})$ will "count," with signs the number of soliton paths with homology class γ_{ij}

2. Interfaces & Formal Parallel Transport

We also recall that we had the notion of an interface $\mathcal{I}(\wp)$ between two 2D defects $\mathbb{S}_{z_1}$ and $\mathbb{S}_{z_2}$.

Here $\wp$ is a continuous path in C from z_1 to z_2.

(2D) BPS states bound to the interface are "framed BPS states" and have "charges"

$$\Gamma_{ij'}(z_1, z_2) = \frac{\{\, \Gamma \mid \partial\Gamma = z_2^{(j')} - z_1^{(i)} \,\}}{\Gamma \sim \Gamma + \partial\sigma}$$

Central charge: $Z_{\gamma_{ij'}} = \int_{\gamma_{ij'}} \lambda$

BPS invt:

$$\underline{\overline{\Omega}}(\wp, \vartheta, \cdot) : \Gamma(z_1, z_2) \longrightarrow \mathbb{Z}$$

Now, to define the formal parallel transport we first introduce the homology path algebra:

$$a \in H_1(\Sigma, \{beg(a), end(a)\})$$

$$a \longrightarrow X_a \quad \text{formal variable}$$

$$X_{a_1} X_{a_2} = \begin{cases} X_{a_1+a_2} & \text{if } end\, a_1 = beg\, a_2 \\ 0 & \text{else} \end{cases}$$

Given framed invts we could define the formal parallel transport to be:

$$F(\wp, \vartheta) = \sum_{a \in \Gamma(z_1, z_2)} \overline{\underline{\Omega}}(\wp, \vartheta, a) X_a$$

$$beg(\wp) = z_1 \qquad end(\wp) = z_2$$

Theorem: $\exists !$ BPS degeneracies

1.) $\overline{\underline{\Omega}}(\wp, \vartheta, a)$ $\forall \wp$ $\forall a \in \Gamma(z_1, z_2)$

2.) $\mu(a)$ $a \in \Gamma(z, z)$ $\forall z \in C$

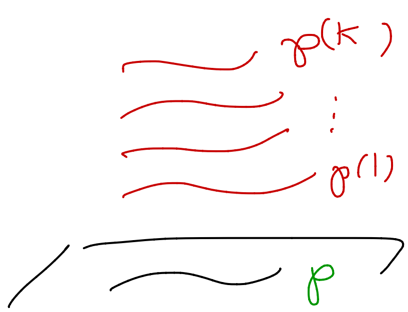
such that $F(\wp, \vartheta)$ satisfies:

A.) Homotopy invariance: $F(\wp_1, \vartheta) = F(\wp_2, \vartheta)$
if $\wp_1 \sim \wp_2$ with fixed endpoints

B.) Homomorphism: $F(\wp_1, \vartheta) F(\wp_2, \vartheta) = F(\wp_1 \circ \wp_2, \vartheta)$

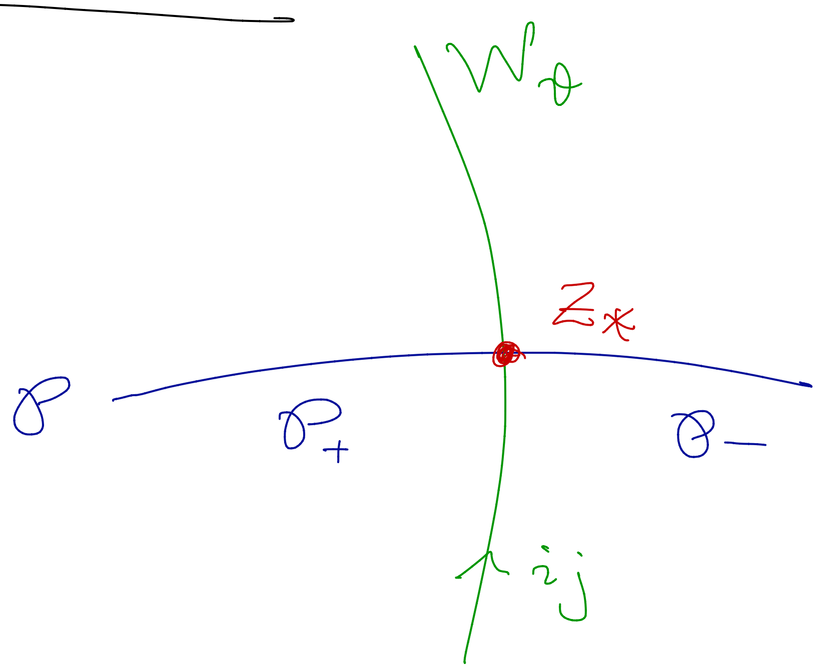
C.) If $\wp \cap \mathcal{W}_\vartheta = \phi$

$$F(\wp, \vartheta) = \sum_{i=1}^{K} X_{\wp^{(i)}} := D(\wp)$$



D.) Detour Rule: - - - -

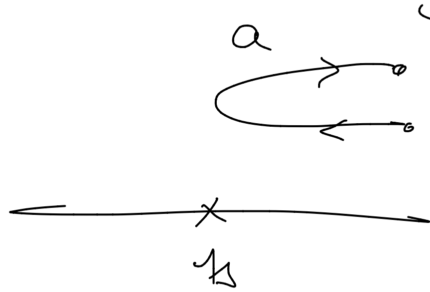
Detour Rule:



$$F(\wp,\vartheta) = D(\wp) + D(\wp_+)\Big(\sum_{a\in\Gamma_{ij}(z_*,z_*)} \mu(a) X_a\Big) D(\wp_-)$$

$$= D(\wp_+) \prod_a \big(1+\mu(a) X_a\big) D(\wp_-)$$

Pf: Build up the $\mu(a)$ and $\overline{\Omega}(\wp,\vartheta,a)$ using these rules starting from the singletons



$\mu(a) = 1$

3. Abelianization, Nonabelianization & True Parallel Transport

Abelianization is well-known in Higgs bundle theory: $E \longrightarrow C$, $\varphi \in \Gamma(K_C \otimes \mathrm{End}\, E)$

$$\mathcal{L} := \ker(\varphi - \lambda) \subset \pi^* E \qquad \pi : \Sigma \to C$$

$$E_z \cong \bigoplus_i \mathcal{L}_{z^{(i)}}$$

Recover bundle from line bundle over the spectral curve

Now we would like to do the converse: Given

1.) Branched cover $\pi : \Sigma \to C$ w/ simple branch points

2.) Complex line bundle $L \to \Sigma$ with flat $GL(1,\mathbb{C})$ connection ∇^{ab} on L

Construct a flat connection on $E = \pi_*(L)$.

The problem with pushing forward ∇^{ab} is that there is an obstruction from monodromy of $\pi : \Sigma \to C$ around branch points.

The naive definition

$$\exp \int_{\wp} \pi_* \nabla^{ab} \overset{?}{=} \sum_{i=1}^{k} \exp \int_{\wp^{(i)}} \nabla^{ab}$$

won't work because $\pi_* \nabla^{ab}$ won't extend over branch points.

<u>Theorem</u>: Given (Σ, L, ∇^{ab}) <u>AND</u> a spectral network W_ϑ there is a complex rank K vector bundle $E_W \to C$ with flat connection ∇_W such that . . . .

1.) On $C - W_\vartheta$ $\quad E_{\mathcal{W}}\big|_{C-W_\vartheta} \cong \pi_*(\mathcal{L})\big|_{C-W_\vartheta}$

2.) For all $z_1, z_2 \in C - W_\vartheta$ and paths on C from z_1, z_2 define

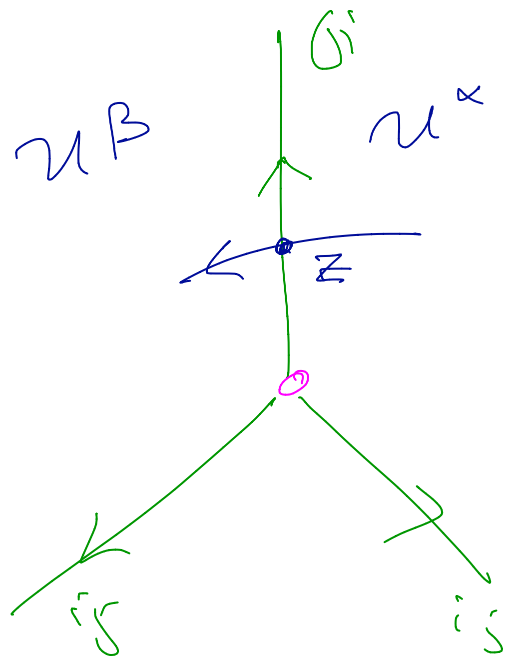
$$\mathbb{F}(p) := \sum_{a \in \Gamma(z_1, z_2)} \overline{\Omega}(\gamma, \vartheta, a)\, \mathcal{Y}_a$$

where $\mathcal{Y}_a \in Hom\left(\bigoplus \mathcal{L}_{z_1^{(i)}}, \bigoplus \mathcal{L}_{z_2^{(j)}}\right)$ is

$$\mathcal{Y}_a = \begin{cases} 0 \quad \text{on } \mathcal{L}_{z_1^{(k)}} \quad k \neq i \\ \exp \int_a \nabla^{ab} \in Hom\left(\mathcal{L}_{z_1^{(i)}}, z_2^{(j)}\right) \\ \text{if } a \in \Gamma_{ij}(z_1, z_2) \end{cases}$$

Then $\mathbb{F}(p)$ is the parallel transport of a flat connection.

Key point in the proof is to define unipotent transition matrices across paths in the spectral network



$$T^{\alpha\beta} = 1 + \sum_{a \in \Gamma(z,z)} \mu(a) \, \mathcal{Y}_a$$

These give coordinates on an open region of $\mathcal{M}_{flat}(C, GL(k), \mathcal{O}_n)$

↑ monodromy data @ punctures

$$\mathcal{M}(\Sigma, GL(1), \{\mu_{n_i}\}) \cong (\mathbb{C}^*)^r$$

$$\cup$$

$$\mathcal{U}_W \xrightarrow[\psi_W]{} \mathcal{M}_{flat}(C, GL(k), \mathcal{O}_n)$$

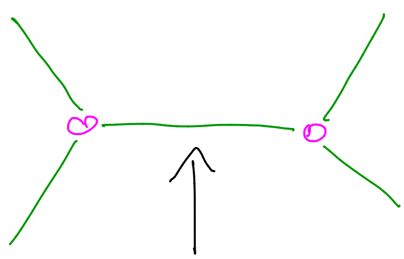
In fact Ψ_W is holomorphic and symplectic:

$$\Psi_W^*\left(\int_C \mathrm{Tr}\, \delta A \, \delta A\right) = \int_\Sigma \delta a \wedge \delta a$$

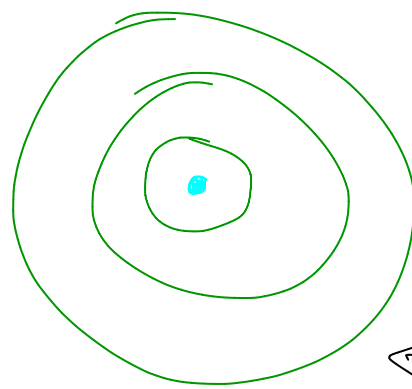
4. BPS States & Morphisms Of Spectral Net's

For generic ϑ the WKB paths in a spectral network begin on branch points and terminate at singular points

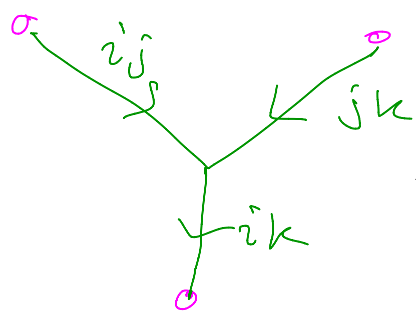
However, for critical values of ϑ we can have "finite WKB curves":



$\Omega(\gamma) = +1$



$\Omega(\gamma) = -2$



$\Omega(\gamma) = +1$

These lift to closed curves on Σ and determine homology classes $\gamma \in H_1(\Sigma, \mathbb{Z})$.

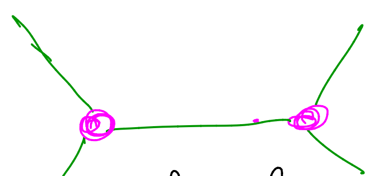
These finite WKB curves are associated to BPS states.

$\gamma \in H_1(\Sigma, \mathbb{Z})$: Charge

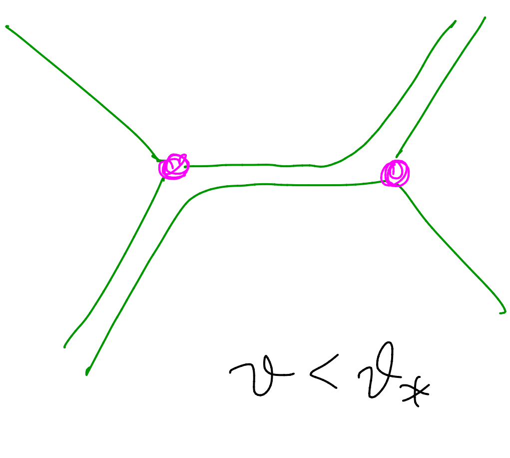
$e^{i\vartheta_*}$: Phase of $\mathcal{N}=2$ central charge

$\Omega(\gamma) \sim$ "counts" the curves

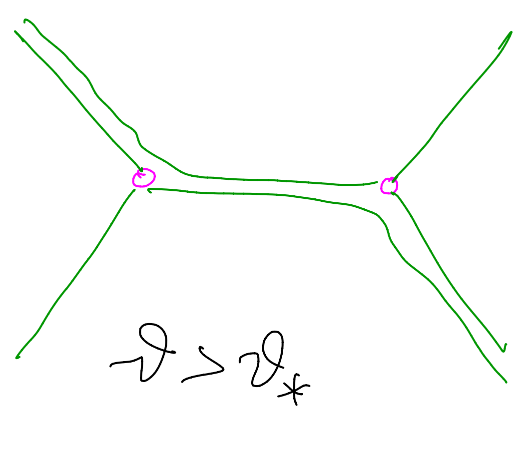
The way the spectral network changes as ϑ varies through ϑ_* gives a wall-crossing formula: e.g.



$\vartheta = \vartheta_*$



$\vartheta < \vartheta_*$



$\vartheta > \vartheta_*$

At ϑ_* we have $\arg Z_{\gamma_*} = \vartheta_*$ for $\gamma_* \in \Gamma$ and one can show

$$\psi_{\mathcal{W}^+}(\nabla^{ab}) = \psi_{\mathcal{W}^-}\left(K_{\beta_0}(\nabla^{ab})\right)$$

Where ∥ transport wrt $K_{\beta_0}(\nabla^{ab})$ is computed by

$$K_{\beta_0} Y_a = \prod_{\gamma_0 : \arg Z_{\gamma_0} = \vartheta_*} \left(1 - Y_{\gamma_0}\right)^{\langle a, \gamma_0 \rangle \Omega(\gamma_0)} Y_a$$

(Note sign. Here $Y_{\gamma_1} Y_{\gamma_2} = (-1)^{\langle \gamma_1, \gamma_2 \rangle} Y_{\gamma_1 + \gamma_2}$)

The parallel transport itself

$$\sum_a \overline{\underline{\Omega}}(\wp, \vartheta, a) Y_a$$

is unchanged $\Rightarrow$ wall-crossing of the framed BPS states

5. Comments & Applications

1.) In the construction of HK metrics on Hitchin moduli space the $\mathcal{Y}_\gamma$ become functions of

$$(u \in \mathcal{B}, \vec{\theta} \in \Gamma \otimes \mathbb{R}/\mathbb{Z}, \zeta)$$

and satisfy an integral equation (formally analogous to the TBA)

2.) $$\nabla_{\mathcal{N}}^{\text{nonab}} = d + \mathcal{A}$$
$$= d + \frac{1}{\zeta}\varphi + A + \zeta\bar{\varphi}$$

So $$(\zeta d + \varphi + \cdots)\Psi = 0$$

for flat sections / parallel transport

$\therefore$ $\zeta \sim \hbar$ and $\zeta \to 0$ asymptotics are closely related to WKB analysis.

The $\mathcal{Y}_\gamma$ have good $\zeta \to 0$ asymptotics

The W_ϑ are then essentially Stokes' lines

3.) In the 4-dimensional Theory we mentioned line defects $L_{\mathcal{R},\wp,\vartheta}$

$\mathcal{R}$ — rep$^{\underline{n}}$ of G

$\wp$ — path on C

ϑ — angle used to construct line defect

Suppose $L_{\mathcal{R},\wp,\vartheta}$ wraps S^1 in $M^{1,2} \times S^1$ and sits at a point in space time.
It is then an operator in the 3D σ-model $M^{1,2} \longrightarrow \mathcal{M}_H$

From 6D origin of class S one argues

$$\langle m | L_{\mathcal{R},\wp,\vartheta} | m \rangle = \mathrm{Tr}_{\mathcal{R}} \left(P\exp \int_{\wp} \mathcal{A} \right)$$

$m \in \mathcal{M}_H$ defines a vacuum in the 3D σ-model and corresponds to flat connection $\mathcal{A}$ for generic complex structure.

Then GMN claim:

$$\langle L_{\mathcal{R},\wp,\vartheta} \rangle = \sum_{\gamma} \overline{\underline{\Omega}}(\mathcal{R},\vartheta,\gamma) Y_{\gamma}$$

4D framed BPS degeneracy

This leads to two further considerations

1.) $\exists$ quantum generalization, and this formula gives an elegant way to derive the KSWCF.

2.) One can also <u>compute</u> $\langle L \rangle$ in complexified FN coordinates and the comparison is quite interesting.